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Long-Term Third-Body Effects via Double Averaging

Roger A. Broucke*
The University of Texas at Austin, Austin, Texas 78712

We study the long-term effects of a third body on a satellite of negligible mass. We have in mind to study the luni-solar effects on an Earth satellite but many other applications can be imagined. We begin with the representation of the disturbing function in an infinite series in Legendre polynomials but we truncate it at the second-degree terms. Our approach consists of a double analytic averaging: with the period of the satellite and with the period of the third body. We concentrate on using the two important integrals (energy and angular momentum) to discuss and classify properties of the perturbed orbits of the satellite. For inclinations below 39 deg, the perigee of the orbits always circulates. There are circular and elliptic orbits but the eccentricity does not vary much. For the inclinations above 39 deg, the circular orbits are unstable and the eccentricities can increase rapidly but we have the appearance of new stable orbits: two elliptic frozen orbits with constant eccentricity and fixed perigee location, at either 90 or 270 deg.

Nomenclature

a	=	one of the six orbit elements: the semimajor axis
a'	=	semimajor axis of the orbit of the perturbing body
C_z	=	z component of the angular momentum
C_1	=	another form of the z component of the angular momentum
C_2	=	form of the Jacobi integral
D	=	angle $\Omega - M'$
e	=	one of the six orbit elements: the eccentricity
G	=	constant of gravity
h	=	equinoctial element $e\sin(\omega)$
i	=	one of the six orbit elements: the inclination
J	=	Jacobi integral
k	=	equinoctial element $e\cos(\omega)$
M	=	mean anomaly angle of the satellite
M_0	=	one of the six orbit elements: the mean anomaly at epoch $t = 0$
M'	=	mean anomaly angle of the perturbing body
m_0	=	mass of the central body
m'	=	mass of the perturbing planet
n	=	mean motion, = average angular velocity of the satellite
n'	=	mean motion of the perturbing body
$\hat{\boldsymbol{P}}$	=	unit vector pointing in the direction of the periapsis of the satellite
P_{ij}	=	five nonzero Poisson brackets in classical orbit elements
$\hat{\boldsymbol{Q}}$	=	unit vector that is 90 deg ahead of $\hat{P}$ and in the plane of the motion of the satellite
$\mathcal{R}$	=	third-body disturbing function
$\mathcal{R}_0$	=	zero-order term of the disturbing function in its expansion in Legendre polynomials
$\mathcal{R}_2$	=	second-order term of the disturbing function in its expansion in Legendre polynomials
$\boldsymbol{r}$	=	radius vector of the satellite, = distance to the central body
$\boldsymbol{r}'$	=	radius vector of the perturbing body
$\hat{\boldsymbol{r}}'$	=	unit vector along the radius vector of the perturbing body
t	=	time
x, y, z	=	rectangular coordinates of the satellite
x', y'	=	rectangular coordinates of the perturbing body (There is no z' coordinate because the orbit of the perturbing body is used to define the $x - y$ plane of the frame of reference.)
α	=	dot product of $\hat{\boldsymbol{P}} \cdot \hat{\boldsymbol{r}}'$
β	=	dot product of $\hat{\boldsymbol{Q}} \cdot \hat{\boldsymbol{r}}'$
γ	=	constant $0.6 = \frac{3}{5}$
δ	=	discriminant of the quadratic equation
μ	=	Gm_0, the gravity constant of the central body
μ'	=	gravity constant of the perturbing body
ϕ	=	latitude, measured on a meridian, from the equator
Ω	=	one of the six orbit elements: the longitude of ascending node
ω	=	one of the six orbit elements: the argument of the periapsis

Introduction

WE describe the problem of the third-body perturbations on a satellite in a simplified approximate model that has been double averaged, over the short satellite period as well as with respect to the distant perturbing body. The perturbing body is in a circular orbit in the (x, y) plane.

The double-averaged problem we are treating is completely integrable and much has already been written about it, for instance, by Kozai,[1] Lidov,[2] Anderson,[3] Lorell,[4] and Williams,[5] all in the early 1960s. It was already shown by these authors that the solution can be expressed in elliptic integrals.

In the meantime a large number of papers have been published on the subject of averaged third-body perturbations. Averaged Hamiltonians were given by Kovalevsky,[6] Harrington,[7,8] and Lidov and Ziglin.[9,10] As for the Harrington Hamiltonian, see also Ferrer and Osacar.[11] Kinoshita and Nakai[12] published a paper on the Uranus satellites, where both the J_2 effect as well as the solar effects are averaged in a single consistent reference frame. The two papers by Kwok[13,14] also use a common frame of reference. The paper by Collins and Cefola[15] gives a high-order recursive third-body averaging formulation.

One of the principal applications of the present study is to evaluate the effect of the lunar and solar perturbations on high-altitude Earth satellites, especially those in nearly circular orbits. Our conclusion is that such circular orbits are stable (i.e., remain circular) only if the mutual inclination with the perturbing body is less than 39 deg. This critical angle (such that $\cos^2 i = 0.60$) is sometimes called the critical inclination of the third-body perturbations. For all of the orbits with a higher inclination, the eccentricity variation will usually be large, and circular orbits will rapidly become more

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*Emeritus Professor, Department of Aerospace Engineering and Engineering Mechanics. Member AIAA.

and more elliptic. In fact, for 90-deg inclination the eccentricity always evolves from zero to one at such a point that these polar orbits can never survive for very long times. The practical implication is thus that the only useful circular orbits in mission design are those with small inclination. In our simplified model the upper value for the inclination angle is 39 deg. However in a more accurate model (for instance, a model with more terms in the Legendre polynomial expansion) this limiting angle is even less than 39 deg (Ref. 1). This could probably also be verified with a more accurate averaging model, such as the Halphen method.

As for the principal properties of the solutions of this integrable dynamical system, we will mention the important fact that there are basically two important classes of orbits, according to the sign of an important first integral C_2 [defined later in Eq. (24)]. When C_2 is positive, all of the solutions have a circulating perihelion: the argument of perigee is always increasing. However, when C_2 is negative the perigee always oscillates around an average value of either 90 or 270 deg. We call these orbits libration orbits; their argument of periapsis ω never passes through 0 or 180 deg.

The integral C_2 is one of the two important integrals of the problem, the other being the normalized z component of the angular momentum $C_1 = (1 - e^2)\cos^2 i$. When C_1 is larger than 0.6, we actually have stable circular orbits. On the other hand, when C_1 is less than 0.6 we have a family of frozen orbits with constant eccentricity e and inclination $\cos^2 i = e^2 + 0.6$, as well as fixed position of the perigee at $\omega = 90$ or 270 deg.

The following two sections give the disturbing function of the third-body perturbations, averaged once, over the period of the satellite (the short period). This material was taken from Kaufman and Dasenbrock,[16] and therefore no derivations are given in our present paper.

Single-Averaged Third-Body Disturbing Function

We recall here the general form of the third-body disturbing function truncated after the second-order term in the Legendre polynomical expansion, under the assumption that the disturbing body is far from the perturbed body ($r' \gg r$), the disturbing body being outside the orbit of the perturbed body:

$$\mathcal{R} = \mu'(a^2 n'^2/2)(a'/r')^3\left[\left(1 + \tfrac{3}{2}e^2\right)\mathcal{R}_0 + (5e^2/2)\mathcal{R}_{c2}\right] \quad (1)$$

where

$$\mathcal{R}_0 = \tfrac{3}{2}(\alpha^2 + \beta^2) - 1, \qquad R_{c2} = \tfrac{3}{2}(\alpha^2 - \beta^2) \quad (2)$$

Here $\alpha = (\hat{\boldsymbol{P}} \cdot \hat{\boldsymbol{r}}')$ and $\beta = (\hat{\boldsymbol{Q}} \cdot \hat{\boldsymbol{r}}')$.

Special Case of a Circular Disturbing Body Orbit

We will now limit ourselves to the special case where the perturbing body is in a circular orbit with given radius $a' = \boldsymbol{r}'$ in the (x, y) plane. In this case the problem has cylindrical symmetry around the z axis in the sense that the mean anomaly M' of the disturbing body enters the problem only through the difference $D = \Omega - M'$.

The two dot products α and β can then be written as

$$\alpha = +\cos\omega\cos D - \cos i\sin\omega\sin D$$

$$\beta = -\sin\omega\cos D - \cos i\cos\omega\sin D \quad (3)$$

assuming that

$$x' = a'\cos M', \qquad y' = a'\sin M', \qquad z' = 0 \quad (4)$$

Substitution of α and β in the two expressions [Eq. (2)] gives

$$\mathcal{R}_0 = \tfrac{3}{2}(\cos^2 D + \cos^2 i\sin^2 D) - 1 \quad (5)$$

$$\mathcal{R}_{c2} = \tfrac{3}{2}[(\cos^2 D - \cos^2 i\sin^2 D)\cos 2\omega - \cos i\sin 2D\sin 2\omega] \quad (6)$$

In the derivation of the equations of motion, we, of course, need the partial derivatives of these two expressions with respect to the orbit elements (i, ω, Ω). Therefore we give here these six partial derivatives:

$$\frac{\partial\mathcal{R}_0}{\partial i} = -\frac{3}{2}\sin^2 D\sin 2i, \qquad \frac{\partial\mathcal{R}_0}{\partial\omega} = 0$$

$$\frac{\partial\mathcal{R}_0}{\partial\Omega} = -\frac{3}{2}\sin 2D\sin^2 i \quad (7)$$

$$\frac{\partial\mathcal{R}_{c2}}{\partial i} = +\frac{3}{2}\cos 2\omega\sin 2i\sin^2 D + \frac{3}{2}\sin 2\omega\sin 2D\sin i$$

$$\frac{\partial\mathcal{R}_{c2}}{\partial\omega} = -3\sin 2\omega\cdot(\cos^2 D - \cos^2 i\sin^2 D) - 3\cos 2\omega\sin 2D\cos i$$

$$\frac{\partial\mathcal{R}_{c2}}{\partial\Omega} = -\frac{3}{2}\cos 2\omega\sin 2D\cdot(1 + \cos^2 i) - 3\sin 2\omega\cos 2D\cos i \quad (8)$$

Double-Averaged Problem

In this section we begin the study of the double-averaged problem, where we perform the average of the disturbing function (1) with respect to the mean anomaly M' of the perturbing body, taking into account the simplifying circular orbit assumptions of the preceding section.

We will use the symbol $\langle\,\rangle$ to indicate the average with respect to the mean anomaly M'. We have the following elementary averages:

$$\langle\cos M'\rangle = 0, \qquad \langle\sin M'\rangle = 0$$

$$\langle\cos D\rangle = 0, \qquad \langle\sin D\rangle = 0 \quad (9)$$

Using the two trigonometric identities

$$\cos^2 D = \tfrac{1}{2}(1 + \cos 2D), \qquad \sin^2 D = \tfrac{1}{2}(1 - \cos 2D) \quad (10)$$

also allows us to write the two following averages:

$$\langle\cos^2 D\rangle = \langle\sin^2 D\rangle = \tfrac{1}{2} \quad (11)$$

The average of the two terms of the disturbing function can thus easily be obtained:

$$\langle\mathcal{R}_0\rangle = \tfrac{1}{4}(3\cos^2 i - 1), \qquad \langle R_{c2}\rangle = \tfrac{3}{4}\sin^2 i\cos 2\omega$$

which gives the following final expression for the double-averaged third-body disturbing function:

$$\langle\mathcal{R}\rangle = \mu'(a^2 n'^2/16)[(2 + 3e^2)(3\cos^2 i - 1) + 15e^2\sin^2 i\cdot\cos 2\omega] \quad (12)$$

We notice an important consequence of the averaging process: the longitude of the ascending node Ω has been completely eliminated. Therefore the problem has basically only three variables: e, i, and ω. The principal consequence of the absence of the angle Ω from the disturbing function is that the z component of the angular momentum will be a constant.

To write down the equations of motion, we of course need the partial derivatives of the disturbing function. The only nonzero partial of $\langle\mathcal{R}_0\rangle$ is

$$\frac{\partial\langle\mathcal{R}_0\rangle}{\partial i} = -\frac{3}{4}\sin 2i \tag{13}$$

The two only nonvanishing partials of $\langle\mathcal{R}_{c2}\rangle$ are

$$\frac{\partial\langle\mathcal{R}_{c2}\rangle}{\partial\omega} = \frac{-3}{2}\sin^2 i\sin 2\omega \tag{14}$$

$$\frac{\partial\langle\mathcal{R}_{c2}\rangle}{\partial i} = +\frac{3}{4}\cos 2\omega\sin 2i \tag{15}$$

The four equations of motion will be

$$\frac{\mathrm{d}e}{\mathrm{d}t} = +P_{25}\frac{\partial\langle\mathcal{R}\rangle}{\partial\omega}, \qquad \frac{\mathrm{d}\omega}{\mathrm{d}t} = -P_{25}\frac{\partial\langle\mathcal{R}\rangle}{\partial e} - P_{35}\frac{\partial\langle\mathcal{R}\rangle}{\partial i}$$
$$\frac{\mathrm{d}i}{\mathrm{d}t} = +P_{35}\frac{\partial\langle\mathcal{R}\rangle}{\partial\omega}, \qquad \frac{\mathrm{d}\Omega}{\mathrm{d}t} = -P_{36}\frac{\partial\langle\mathcal{R}\rangle}{\partial i} \tag{16}$$

where the following three Poisson brackets are used:

$$P_{25} = \frac{-\sqrt{1-e^2}}{na^2 e}, \qquad P_{36} = \frac{-1}{na^2\sqrt{1-e^2}\sin i}$$
$$P_{35} = -\cos i\, P_{36} \tag{17}$$

Substituting these brackets in the preceding equations of motion gives us the following result:

$$\frac{\mathrm{d}e}{\mathrm{d}t} = \frac{15\mu' n'^2 e\sqrt{1-e^2}}{8n}\sin^2 i\sin 2\omega$$
$$\frac{\mathrm{d}i}{\mathrm{d}t} = \frac{-15\mu' n'^2 e^2}{16n\sqrt{1-e^2}}\cdot\sin 2i\,\sin 2\omega$$
$$\frac{\mathrm{d}\omega}{\mathrm{d}t} = \frac{3\mu' n'^2}{8n\sqrt{1-e^2}}\cdot[(5\cos^2 i - 1 + e^2) + 5(1-e^2-\cos^2 i)\cos\omega]$$
$$\frac{\mathrm{d}\Omega}{\mathrm{d}t} = \frac{3\mu' n'^2\cos i}{8n\sqrt{1-e^2}}\cdot[5e^2\cos 2\omega - 3e^2 - 2]$$
$$\frac{\mathrm{d}M_0}{dt} = \frac{-\mu' n'^2}{8n}[(3e^2+7)(3\cos^2 i - 1)$$
$$+ 15(1+e^2)\sin^2 i\cos^2\omega] \tag{18}$$

We immediately see that the right-hand sides of the equations of motion contain essentially only three variables: e, i, and ω. This is because two elements are absent: the mean anomaly because of the averaging and the ascending node because of the cylindrical symmetry. The last element, the semimajor axis a, is constant. Consequently, we have a system of only three simultaneous ordinary differential equations.

The fourth equation is completely uncoupled from the three first ones. In other words, the longitude of the ascending node Ω is driven by the variables (e, i, ω) without influencing these three variables.

The fifth equation, for the mean anomaly at epoch M_0, is included here, although it probably does not have much meaning anymore because the mean anomaly was removed from the problem in the averaging process. However, some authors have used the equation for M_0 in order to recover a mean anomaly $M = M_0 + nt$.

We will later describe the properties of the solutions of the preceding system (18). For the time being we only mention that there are circular solutions; when e is initially zero, it stays zero for all times. We also see that these circular solutions have constant inclination. On the other hand, when $\sin(i)$ is initially zero it remains zero for all times. Thus, there exist elliptic equatorial solutions. These solutions all have constant eccentricity.

Two First Integrals of the System

Besides the fact that the semimajor axis a remains constant, the solutions of the present double-averaged system also have two other important first integrals that are essentially the total energy (Hamiltonian) and the z component of the angular momentum.

As for the angular momentum C_z, we verify immediately from the two first equations of motion (18) that

$$e\cos i\cdot\dot{e} + (1-e^2)\sin i\cdot\dot{i} = 0 \tag{19}$$

which is equivalent to

$$\frac{\mathrm{d}}{\mathrm{d}t}[(1-e^2)\cos^2 i] = 0 \tag{20}$$

We will write this result in the form

$$(1-e^2)\cos^2 i = C_1 \tag{21}$$

where C_1 is thus the z component of the angular momentum squared. We see that C_1 can vary only from 0 to 1. A consequence of this integral is that e and i are always required to vary in opposite directions (in the first quadrant).

As for the total energy and the disturbing function, they are constant because they do not explicitly contain the time t. We have thus from the formula (12) for the disturbing function $\mathcal{R}$:

$$\tfrac{1}{2} + \tfrac{3}{4}e^2 - \tfrac{3}{4}\sin^2 i + \tfrac{3}{4}e^2\sin^2 i - \tfrac{15}{4}e^2\sin^2 i\sin^2\omega = C \tag{22}$$

However, we will combine this integral C with the integral C_1 in order to derive a new integral that is considerably simpler. The combination $C + 3C_1/4$ gives

$$\tfrac{3}{2}e^2 - \tfrac{15}{4}e^2\sin^2 i\sin^2\omega = C'' \tag{23}$$

or

$$e^2\left(\tfrac{2}{5} - \sin^2 i\sin^2\omega\right) = C_2 \tag{24}$$

Considering that $(\sin i\sin\omega)^2$ can vary only from 0 to 1, we conclude that

$$\tfrac{-3}{5} \le C_2 \le \tfrac{-2}{5} \tag{25}$$

We will consider C_1 and C_2 as the two fundamental first integrals of the problem.

It is easy to determine the upper limit of C_2 for a given value of C_1. It corresponds to the lower limit of $\sin^2 i\,(=0)$ or $\cos^2 i = +1$. In this case we have from the two preceding equations

$$C_1 = 1 - e^2, \qquad C_2 = \tfrac{2}{5}e^2 \tag{26}$$

which gives, after elimination of e^2,

$$C_2 = \tfrac{2}{5}(1 - C_1) \tag{27}$$

This is a straight line in the (C_1, C_2) plane, corresponding to equatorial elliptic orbits (i.e., with zero inclination) with constant eccentricity:

$$e = \sqrt{1 - C_1} \tag{28}$$

By using the two basic first integrals, we can reduce the order of the system from 3 to 1, which means that the system is completely integrable. A rather simple approach consists in eliminating the inclination i with the use of the C_1 integral, so that we are left with a second-order system whose solutions can be plotted in either the (ω, e) plane or the (k, h) plane. Here, k and h represent the so-called equinoctial variables that we will often use to represent and classify the different orbits of this sytem: $k = e\cos(\omega)$ on the horizontal axis and $h = e\sin(\omega)$ on the vertical axis.

In this coordinate system all of the orbits are symmetric with respect to both the horizontal and the vertical axis. Strictly speaking, we would thus only have to compute and integrate the orbits in the first quadrant, where ω is from 0 to 90 deg.

We will also see later that there are some elliptic fixed points on these diagrams, on the vertical axis, thus corresponding to $\omega = 90$ or 180 deg, with $\cos(2\omega) = -1$.

Another important property is that there is an important bifurcation of solutions at $e = 0$ and $\cos(i) = \sqrt{\frac{3}{5}}$. This corresponds thus to $C_1 = \frac{3}{5} = 0.60$. The corresponding inclination is $i = 0.684719203$ radians or 39.23152048 deg. This angle is sometimes called the third-body critical inclination.

The expression $\sin(i)\sin(\omega)$ that is present in the equations of motion has a simple geometric meaning: it is equal to $\sin(\phi)$, where ϕ is the latitude of the satellite when it passes its periapsis point in its orbit.

Orbits with Given C_1 and C_2

In many applications we will have to compute orbits with a given value of C_1 and C_2. Therefore it is necessary to be able to solve the two equations (21) and (24) for e and i for instance. To do this, we first eliminate the inclination i with

$$-\sin^2 i = C_1/\epsilon - 1 \tag{29}$$

where we have defined $\epsilon = 1 - e^2$. After some simple manipulations Eq. (24) for C_2 reduces to a quadratic equation in ϵ, (with $\gamma = 0.6$):

$$[\sin^2\omega - (1-\gamma)]\epsilon^2 + \left[(1-\gamma) - (1+C_1)\sin^2\omega - C_2\right]\epsilon + \left[C_1 \sin^2\omega\right] = 0 \tag{30}$$

In many applications we use the initial value $\omega = 90$ deg for the orbits that we integrate. So, in the case $\sin\omega = 1$, we have the following simpler equation:

$$\gamma\epsilon^2 - (C_1 + C_2 + \gamma)\epsilon + C_1 = 0 \tag{31}$$

This quadratic equation in ϵ can also be written as a quadratic in e^2:

$$\gamma e^4 + (C_1 + C_2 - \gamma)e^2 - C_2 = 0 \tag{32}$$

In both cases only the roots between 0 and 1 are acceptable, of course.

It is worthwhile to examine the discriminant of this quadratic equation (31) in ϵ:

$$\delta = (C_1 + C_2 + \gamma)^2 - 4\gamma C_1 \tag{33}$$

The double roots correspond to the values $\delta = 0$. One of the cases corresponds to the values $C_1 = 0$ and $C_2 = -\gamma$. This is the point E on the diagram of Fig. 1 described in the next section. Another such point corresponds to $C_2 = 0$ and $C_1 = \gamma$, $[(2\gamma)^2 - 4\gamma\gamma = 0]$. This is the point D on the same diagram. It corresponds to the bifurcation of the circular solutions (the critical inclination) described in the following section.

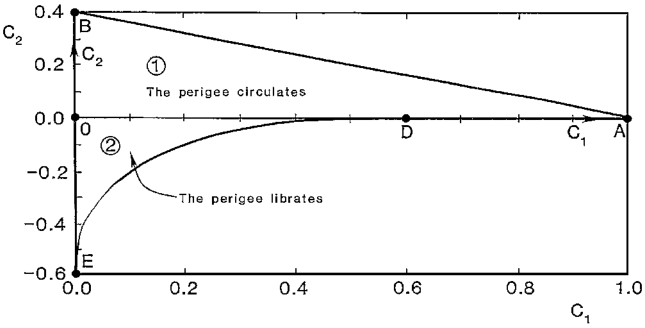

Fig. 1 This fundamental (C_1, C_2) diagram allows us to make a complete discussion of all of the possible types of orbits. The upper half $C_2 > 0$ corresponds to a circulating perigee: ω goes all of the way around, from 0 to 360 and so on. The lower half $C_2 < 0$ corresponds to orbits with a librating perigee: ω oscillates around 90 or 270 deg. The boundary between the two regions, the horizontal axis ODA, $C_2 = 0$, corresponds to exactly circular orbits as is seen in Eq. (24) defining C_2. The extreme right ($C_1 = 1$) corresponds to circular equatorial orbits, while the extreme left side ($C_1 = 0$) corresponds to rectilinear and polar orbits. C_1 is the z component of angular momentum.

Equilibrium Solutions (Frozen Orbits)

We will show here that there exist special solutions corresponding to constant values for all three of the orbit elements e, i, and ω. We need therefore $\dot{e} = \dot{i} = \dot{\omega} = 0$. A close examination of the equations of motion shows that the three right-hand sides can be simultaneously zero only where $\sin 2\omega = 0$ and $\cos 2\omega = -1$. In other words, the argument of the peripasis ω must be 90 or 270 deg. The equation for ω then gives

$$(5\cos^2 i - 1 + e^2) - 5(1 - e^2 - \cos^2 i) = 0 \tag{34}$$

which reduces to a relation between the constant values of e and i,

$$e^2 = 1 - \tfrac{5}{3}\cos^2 i \tag{35}$$

As e must be positive, we must also have

$$\cos^2 i < \tfrac{3}{5} \tag{36}$$

In other words, these equilibrium solutions do not exist for small inclinations. The lowest possible inclination corresponds to the circular orbit ($e = 0$) with inclination= 39.23152 deg (such that $\cos^2 i = 0.6 = \gamma$).

Using the first integral C_1 shows us that these equilibrium solutions exist only for $0 < C_1 < 0.6$. In fact, if we set $\sin^2\omega = 1$, we have the following three relations: $e^2 = 1 - \frac{5}{3}\cos^2 i$, $C_1 = (1-e^2)\cos^2 i$, and

$$C_2 = e^2\left(\cos^2 i - \tfrac{3}{5}\right) \tag{37}$$

Eliminating e^2 and $\cos^2 i$ in these three relations gives us a single relation between C_1 and C_2:

$$C_2 = -\gamma - C_1 + 2\sqrt{\gamma C_1} = -\gamma\left(1 - \sqrt{C_1/\gamma}\,\right)^2 \tag{38}$$

where $\gamma = 0.6$. It turns out that for any value of C_1 (less than γ) the lowest possible value of C_2 is the one given by the preceding formula.

Two Principal Classes of Solutions

In the preceding section we discovered an equation for the lowest possible value of C_2 for a given value of C_1. We will first show here that a similar upper value of C_2 can also be determined. It corresponds to the limiting situation $\cos^2 i = 1$ (the equatorial orbits). The two first integrals reduce to $1 - e^2 = C_1$ and $C_2 = 2e^2/5$. Elimination of e^2 gives the required Eq. (27), $C_2 = \frac{2}{5}(1 - C_1)$.

In the (C_1, C_2) plane this equation gives thus a straight line connecting the points (1, 0) and (0, $\frac{2}{5}$).

We conclude that all of the possible solutions will be represented by a point inside the quasitriangle ABEDA shown in Fig. 1. Only the points inside the triangle ABEDA are admissible values of C_1 and C_2. Consequently, there will essentially be two and only two basic classes of orbits, according to the sign of C_2.

The principal characteristics of the two classes of orbits can be summarized as follows:

Region 1: Orbits with $C_2 > 0$

These orbits all have circulating periapsis. The periapsis is always advancing. (ω is always increasing.) The eccentricity and inclination oscillate and reach their extreme values when the orbits cross the horizontal or vertical axis in the (k, h) plane ($\omega = 0$, 90, 180, or 270 deg). These orbits exist for all values of C_1: from the circular orbit with $C_1 = 1$ all of the way down to 0.

Region 2: Orbits with $C_2 < 0$

These orbits have a periapsis orbit that librates around the average value of 90 or 270 deg, where both e and i reach their extreme values. These orbits exist only for C_1 between 0 and $\frac{3}{5}$. The periapsis librates between two extreme values, symmetric with respect to the vertical axis $\omega = 90$ or 270 deg. It can never go through 0 or 180 deg.

Boundary Orbits

In the present section we will give a separate description of the orbits corresponding to the segments AB, AD, DO, ED, OB, and OE in the triangle ABEDA of Fig. 1.

Segment AB

The orbits correspond to $2(1-C_1)=5C_2$, as has been seen before. Substituting for C_1 and C_2 the expressions (21) and (24) gives the following relation between the three orbit elements e, i, ω:

$$\sin^2 i \cdot \left[\tfrac{5}{2}e^2 \sin^2 \omega + (1-e^2)\right] = 0$$

Normally only the first of the two factors is zero: $\sin^2 i = 0$ or $i = 0$ or 180 deg. These are thus the planar (equatorial) elliptic orbits, with constant eccentricity $e^2 = 1 - C_1$ and with increasing ω (advancing periapsis).

We note that the second factor can only be zero if each of the two terms vanishes: $e = 1$ and $\sin^2 \omega = 0$. This corresponds to point B, where $C_1 = 0$ and $C_2 = \frac{2}{5}$.

Curve ED

This curve has been well described before. It corresponds to the equilibrium solutions with constant e and i, related by Eq. (35), $e^2 = 1 - \frac{5}{2}\cos^2 i$, as well as a constant $\omega = 90$ or 270 deg. They only exist for $C_1 < \frac{3}{5}$. Note that these fixed points are of the elliptic type: they are surrounded by small ovals. Therefore, these solutions are called stable beacuse an initially small perturbation from the fixed point will remain small.

Segment DA ($C_2 = 0$)

This segment corresponds to stable circular orbits; $e = 0$.

Segment OD ($C_2 = 0$)

This segment corresponds to the configuration $\sin^2 i \sin^2 \omega = \frac{2}{5}$. These orbits are the separatrices between the two major classes 1 and 2 of orbits described in the preceding section. The eccentricity e tends towards zero or away from zero.

Segment BE ($C_1 = 0$)

As a consequence of the angular momentum integral, this segment must correspond to polar orbits ($\cos^2 i = 0$) or to rectilinear orbits. For all polar orbits the eccentricity tends to 1.

Point A ($C_1 = 1$; $C_2 = 0$)

This point corresponds to $e = 0$ and $\sin^2 i = 0$: the circular equatorial orbit.

Point B ($C_1 = 0$; $C_2 = \frac{2}{5}$)

This point represents the rectilinear orbit with $e = 1$ and with arbitrary inclination as well as $\sin^2 \omega = 0$.

Point D ($C_2 = 0$; $C_1 = \frac{3}{5}$)

This is the bifurcation point ($e = 0$ and $\cos^2 i = \frac{3}{5}$: $i = 39.23152$ deg). The periapsis ω is undefined. It is at this point that the second class of orbits (with librating periapsis) first appears (in the order of decreasing values of C_1, from 1 to 0). In other words, this is the first value of C_1 where orbits with negative C_2 are possible.

The 39.23152-deg critical inclination has been discussed by several authors. It is already in Kozai's paper,[1] which is concerned solely with the solar system and the asteroid belt. Just about all of the asteroids have an inclination less than roughly 39 deg in the ecliptic frame.

Point E ($C_1 = 0$; $C_2 = -\frac{3}{5}$)

This point represents polar orbits ($\cos^2 i = 0$) with arbitrary eccentricity and ω such that $e^2(5\sin^2\omega - 2) = 3$. A special case is the rectilinear orbit with $e = 1$ and $\sin\omega = \pm 1$.

Point O

This can represent any one of the following three types of orbits ($C_1 = C_2 = 0$): 1) circular polar orbits with initially undefined periapsis ω ($e = 0$, $\cos^2 i = 0$), 2) polar orbits with arbitrary eccentricity and with $\sin^2 \omega = \frac{2}{5}$, and 3) rectilinear orbits with arbitrary inclination ($e = 1$) and ω such that

$$\sin^2 i \sin^2 \omega = \tfrac{2}{5}$$

Contour Plots of the Solutions

It is possible to display the solutions in the (ω, e) plane or in the (k, h) plane without actually integrating the equations of motion. Instead we draw level curves with the use of the two first integrals C_1 and C_2.

For instance, we can express C_2 as a function of e, ω, and C_1:

$$C_2 = e^2\left\{0.4 - \left[1 - C_1/(1-e^2)\right]\sin^2\omega\right\} \tag{39}$$

This equation will give different curves C_2, as a function of e and ω, all corresponding to the same constant value of C_1.

In a similar fashion we can also express C_2 as a function of h and k:

$$C_2 = 0.4(h^2 + k^2) - \left[1 - C_1/(1 - h^2 - k^2)\right]h^2$$

Discussion of the Orbits in the (k, h) Plane

We will comment here on the properties of the orbits in terms of the space phase plots. We refer to the (k, h) diagram (Fig. 2). In these diagrams the circular orbits $h = k = 0$ are at the origin that is called a fixed point. In general, the radius vector is the magnitude of the eccentricity, and the polar angle is the argument of the perigee.

As for the definition of stability, it is generally accepted in mathematics and dynamics that a fixed point is stable, if it is of the elliptic type. That means that the curves around it are small ellipses or ovals. That also means that a small initial perturbation will not grow too fast. So, eccentricity would not grow too fast.

A fixed point is unstable if it is of the hyperbolic type. That means that the curves close to it are hyperbolas. That also means that a small initial perturbation will grow fast. So, eccentricity would increase fast. This would especially go very fast if we happen to be on an asymptote of the hyperbola. This is called a stable-unstable manifold in the modern dynamics literature, and the transition from stable to unstable circular orbits is called a pitchfork bifurcation.

In terms of the classification in Fig. 1, on the segment DA, where i is less than 39 deg, circular orbits are stable because the only possible nearby orbits are above the line DA, which are the ovals with small variations in e.

On the other hand, on the segment OD, where i is larger than 39 deg, they are unstable because there are possible nearby orbits below the line OD, which give hyperbolas with large variations in e.

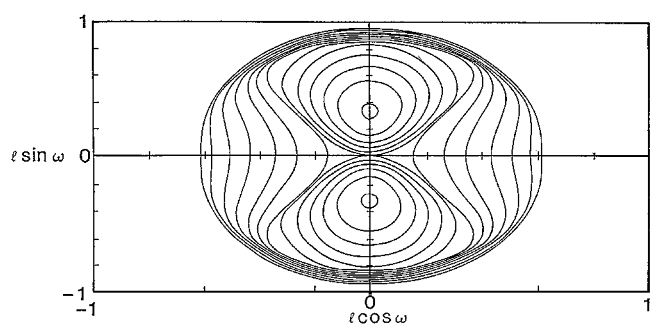

Fig. 2 This figure shows a typical phase diagram. The x axis represents $e\cos\omega$ and the y axis $e\sin\omega$. The curves give thus the evolution of the eccentricity e as well as the perigee location ω. Here i is greater than 39 deg. It shows a hyperbolic point at the center $e = 0$, which is the unstable circular orbit. It also shows the two elliptic fixed points, which are the two stable frozen orbits. One is at $\omega = 90$ deg, and the other is at $\omega = 270$ deg. This is the generic situation in region 2; ($C_2 < 0$) of the fundamental (C_1, C_2) diagram of Fig. 1. The two different kinds of orbits are shown here, with circulating as well as with librating perigee.

If we take a point just below the segment OD, even by an infinitesimal amount, we obtain a new orbit whose eccentricity increases very fast, possibly up to 1.0 (Fig. 2). When we talk about hyperbolas here, we mean it in the (k, h) diagram. The orbits themselves are of course elliptic.

The definition of stable orbits being that there must be small ovals around the fixed points in the (k, h) plane also applies to the frozen orbits, which are thus stable. They are the fixed point at the center of two families of small ovals in the phase diagram at 90 and 270 deg.

Verification of the Numerical Integration

It is well known that in all cases where the perturbing body is in a circular orbit (whether we average or not) the component of the satellite's angular momentum perpendicular to the perturbing body orbit remains constant. This is the Jacobi integral. It is also the Hamiltonian in the corotating frame of reference. It can be written in the form

$$J = -Gm_0/2a - \omega'\sqrt{Gm_0a(1-e^2)} - \mathcal{R}$$

In the single-averaged model the semimajor axis a is also constant. As we have seen in the preceding sections, in the double-averaged model the disturbing function $\mathcal{R}$ remains constant, as well as the polar angular momentum

$$C_z = \sqrt{\mu a(1-e^2)} \cdot \cos i$$

We use three of these integrals a, C_z, and $\mathcal{R}$ to derive the two more simple integrals C_1 and C_2.

Conclusions

We performed a double averaging to the third-body perturbations in order to eliminate all the periodic terms. The remaining dynamical system is integrable, and the types of solutions were discussed in a very simple geometric diagram (Fig. 1) based on two first integrals. It appears that a critical inclination of about 39 deg plays a rather important role.

We have stable circular orbits as well as elliptic orbits with circulating periapsis for i less than 39 deg. At that point the circular orbits become unstable, but two stable elliptic frozen orbits are born. When i is greater than 39 deg, the frozen orbits are surrounded by elliptic orbits with librating perigee and fairly small eccentricity variations.

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